Rigorous solution of Quantum Ising model (S=1)  
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Zhihua Yang (杨志华)

ZheJiang University, ZheJiang Institute of Modern Physics

Collaborators: Prof. Tao Xiang, Institute of Physics  
Dr. Liping Yang, Institute of Theoretical Physics  
Prof. Jianhui Dai, ZheJiang University
Outline

• **Introduction**: Quantum spin system of low dimensions

• Methods

• Results

• Conclusion
Spin system of low dimensions

- Quantum Spin system
  - Spin freedom
  - Localized spin

Quantum Lattice model

- XXZ model
- Heisenberg model
- XY model
- Ising model

Heisenberg

Bethe

Lieb

Ising
The Hamiltonian of $S=1$ Quantum Ising Model

- $S=1$ Quantum Ising Model with single ion anisotropies
  Solved exactly with hole decomposition and recursion method

$$H = \sum_{j=1}^{L} \left[ J S_j^z S_{j+1}^z - 2 D_x (S_j^x)^2 - D_z (S_j^z)^2 \right]$$

Exchange coupling

$J>0$, Antiferromagnetic
$J<0$, Ferromagnetic

Single ion anisotropy

Crystal field

$D_x = 0$ Blume-Capel Model
The Two Properties-I

• Good Quantum number --- Holes:

\[ \hat{N}_0 = L - \sum_{j=1}^{L} (S_j^z)^2 \]

\[ [\hat{N}_0, H] = 0 \]

\[ \mathcal{H} = \sum_{N_0=0}^{L} \oplus \mathcal{H}_{N_0} \]

-1,0,+1

$L$ --- The length of the chain

Corresponding subplace -- sector
The Two Properties-II

- Decoupled

\[(S^x_i)^2 | -1 \rangle = 2 |1 \rangle + 2 | -1 \rangle \]
\[(S^x_i)^2 | 1 \rangle = 2 |1 \rangle + 2 | -1 \rangle \]
\[(S^x_i)^2 | 0 \rangle = |0 \rangle \]

\[\frac{1}{\sqrt{2}} (|1 \rangle + | -1 \rangle) \]
\[\frac{1}{\sqrt{2}} (|1 \rangle - | -1 \rangle) \]

\[(S^x_j)^2 = \frac{1}{4} (S^+_j S^+_j + S^+_j S^-_j + S^-_j S^+_j + S^-_j S^-_j) \]
Outline

• Introduction

• **Methods**: Jordan-Wigner transformation, Hole-Decomposition scheme, Recursion method

• Results

• Conclusion
Mapping onto Transverse Ising Model

• Nonhole sector

\[
S^+_i S^+_i \mid -1 \rangle = 2 \mid 1 \rangle \sim 2 \sigma_i^+ \mid -1 \rangle = 2 \mid 1 \rangle \\
S^-_i S^-_i \mid 1 \rangle = 2 \mid -1 \rangle \sim 2 \sigma_i^- \mid 1 \rangle = 2 \mid -1 \rangle \\
1 - S^-_i S^+_i \mid -1 \rangle = - \mid -1 \rangle \sim \sigma_i^- \mid -1 \rangle = - \mid -1 \rangle \\
S_i^+ S^-_i - \mid 1 \rangle = \mid 1 \rangle \sim \sigma_i^z \mid 1 \rangle = \mid 1 \rangle 
\]

• Hole sectors

\[
H(N_0 = 0, L) = -\sum_{j=1}^{L-1} J \sigma_j^z \sigma_{j+1}^z - \sum_{j=1}^{L} D_x \sigma_j^x \\
H(\{x_i, N_0\}) = \sum_{n=1}^{N_0+1} h(l_n) + N_0 (D_z - D_x) 
\]

Diagonalization

• Jordan-Wigner transformation

Spin operators---fermion operators

\[ s_j^- = \exp[-\pi i \sum_{n<j} c_n^+ c_n] c_j, \quad s_j^+ = c_j^+ \exp[\pi i \sum_{n<j} c_n^+ c_n] \]

Spin system ---- spinless fermion system

\[ H(\{x_i, N_0\}) = \sum_{n=1}^{N_0+1} \sum_{k^{(n)}} \varepsilon(k^{(n)})(\eta_{k^{(n)}}^+ \eta_{k^{(n)}} - \frac{1}{2}) - N_0(D_z - D_x), \]

The energy spectrum

\[ \varepsilon(k^{(n)}) = \pm |D_x| \sqrt{1 + \lambda^2 + 2\lambda \cos k^{(n)}} \quad \lambda = \frac{J}{D_x} \]
Hole Decomposition Scheme

• The partition function can be expressed as,

\[ \mathcal{H} = \sum_{N_0=0}^{L} \bigoplus \mathcal{H}_{N_0} \]

• The partition function can be expressed as,

\[ Z(L) = Z(N_0 = 0) + Z(N_0 = 1) + \cdots + Z(N_0 = L) \]

• Consider all the configuration, take \( N_0 = 2 \) for example
The Recursion Method

• For a given $N_0 - 1$ configuration

\[ z(l_1 = 4) \cdot h \cdot z(l_2 = 3) \cdot h^3 \cdot z(l_3 = 1) \cdot h \cdots z(l_{N_0} = 1) \]

• The recursion initial value: $Z^{(0)}(l) = z(l)$

• The recursion condition: $\sum_{i=1}^{N_0+1} l_i = L - N_0$
The Recursion Method

- We can obtain a $N_0$ configuration

\[
z(l_1 = 4) \cdot h \cdot z(l_2 = 3) \cdot h^3 \cdot z(l_3 = 1) \cdot h \cdot z(l_{N_0} = 1) \cdot h \cdot z(l_{N_0+1} = 3)
\]

- The recursion formula:

\[
Z^{(N_0)}(l) = \sum_{l_{N_0+1} = 0}^{l} Z^{(N_0-1)}(l - l_{N_0+1}) z(l_{N_0+1})
\]

- Example:

\[
Z^{(1)}(l) = \sum_{l_2 = 0}^{l} z(l - l_2) z(l_2)
\]
The Recursion Method

•The recursion partition form:

\[
Z = Tr \exp(-\beta H) = \sum_{N_0=0}^{L} \sum_{l=0}^{L-N_0} h^{N_0-1} Z^{(N_0-1)} (L - l - N_0) h z(l) = \sum_{N_0=0}^{L} h^{N_0} Z^{(N_0)} (L - N_0)
\]

•All the thermodynamics can be exactly solved!

\[
U(T) = \sum_{N_0=0}^{L} h^{N_0} (N_0 + 1) \sum_{l=0}^{L-N_0} \frac{u(l,T) z(l) Z^{(N_0-1)} (L - N_0 - l)}{Z(L)} + (D_z - D_x) N_h
\]

\(U(T)\) can stand for the initial energy, specific heat, susceptibility, spin-spin correlation function, etc.

\(u(l,T)\) represents the corresponding initial energy, specific heat, susceptibility, spin-spin correlation function of the segment with the length \(l\).

\(N_h\) is the thermodynamical average of hole number.
Outline

• Introduction

• Methods

• **Results:** the ground state and the excitation states
  the properties of low temperature

• Conclusion
The ground state and excitations

- $D_z=0$,

$$E_0(N_0, L) < E_0(N_0 + 1, L)$$

- The energy of the GS ($D_z=0$):

$$E_0 = E_0(0, L) = -\frac{1}{2} \sum_k |\varepsilon(k)|$$

- The critical behavior:

$$\lambda = \frac{J}{D_x}$$

\[
\begin{aligned}
|\lambda| < 1, & \text{ paramagnetic regions} \\
|\lambda| = 1, & \text{ critical points} \\
|\lambda| > 1, & \text{ magnetic ordered regions}
\end{aligned}
\]

\[
\begin{aligned}
J > 0, \text{ Ferromagnetic} & \quad \begin{cases} 
\lambda > 0, J > 0, D_x > 0 \\
\lambda < 0, J > 0, D_x < 0
\end{cases} \\
J < 0, \text{ Antiferromagnetic} & \end{aligned}
\]
The ground state and excitations

- The low energy excitation:

Fermionic excitation gap:

$$ \Delta_0 = 2J \left| \frac{1}{|\lambda|} - 1 \right| $$

Hole excitation gap:

$$ \Delta_h^{(N_0)} = E_0(N_0, L) - E_0(0, L) $$
The specific heat

Two peak properties!
Rigorous result VS TMRG result

Low temperature thermodynamics

The competition between hole excitations and fermionic excitations:

\[ T \sim \Delta_h \quad T \sim \Delta_0 \]

The competition between hole excitations and fermionic excitations.
The Quantum critical Properties

The specific heat coefficient

\[ C(T) = \frac{\lambda}{T} \left( 1 + \frac{2}{\lambda} \right) \]

The dimensionless scaled free energy

\[ \Phi(T) = \frac{2J}{\sqrt{|\lambda|}} \left( F(T) - F(0) \right) \]

The ground state when $D_z \neq 0$

- Quantum phase transition of ground state:

$$\Delta_h = E_0(L - 1) - E_0(L) - D_x + D_z$$

$$\Delta_h(D_z = 0) = E_0(L - 1) - E_0(L) - D_x$$

- $D_z > -\Delta_h(D_z = 0)$, the properties of the case is similar with $D_z = 0$

- $D_z < -\Delta_h(D_z = 0)$, the holes domain the system
The thermodynamical properties when $D_z \neq 0$

- The magnetization and the hole number at $\lambda = 1.5$, where $\Delta_h(D_z = 0) = 0.44J$

- From the magnetically ordered region to the hole-condensed region --- The first order
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Conclusion

- The **hole decomposition scheme** to contact the spin-half model with spin-one model, **dimension-independent**;
- The **recursion method** to solve quantum Ising chain with single-ion anisotropy, **model-independent**;
- **All the thermodynamics** quantities are obtained: specific heat, susceptibility, initial energy, …;
- **Hole excitations** and **Fermion excitations**;
- The S=1 quantum Ising model is **the same universality** as S=1/2 transverse Ising chain.
Thank you